

# REAL HYPERSURFACES OF NON - FLAT COMPLEX SPACE FORMS IN TERMS OF THE JACOBI STRUCTURE OPERATOR

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## Abstract

The aim of the present paper is the study of some classes of real hypersurfaces equipped with the condition  $\phi l = l\phi$ , ( $l = R(., \xi)\xi$ ).

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## 0 Introduction.

An  $n$  - dimensional Kaehlerian manifold of constant holomorphic sectional curvature  $c$  is called complex space form, which is denoted by  $M_n(c)$ . The complete and simply connected complex space form is a projective space  $\mathbb{C}P^n$  if  $c > 0$ , a hyperbolic space  $\mathbb{C}H^n$  if  $c < 0$ , or a Euclidean space  $\mathbb{C}^n$  if  $c = 0$ . The induced almost contact metric structure of a real hypersurface  $M$  of  $M_n(c)$  will be denoted by  $(\phi, \xi, \eta, g)$ .

Real hypersurfaces in  $\mathbb{C}P^n$  which are homogeneous, were classified by R. Takagi ([15]). J. Berndt ([1]) classified real hypersurfaces with principal structure vector fields in  $\mathbb{C}H^n$ , which are divided into the model spaces  $A_0$ ,  $A_1$ ,  $A_2$  and  $B$ .

Another class of real hypersurfaces were studied by Okumura [13], and Montiel and Romero [12], who proved respectively the following theorems.

**Theorem 0.1** *Let  $M$  be a real hypersurface of  $\mathbb{C}P^n$ ,  $n \geq 2$ . If it satisfies*

$$g((A\phi - \phi A)X, Y) = 0$$

*for any vector fields  $X$  and  $Y$ , then  $M$  is a tube of radius  $r$  over one of the following Kaehlerian submanifolds:*

- ( $A_1$ ) a hyperplane  $\mathbb{C}P^{n-1}$ , where  $0 < r < \frac{\pi}{2}$ ,*
- ( $A_2$ ) a totally geodesic  $\mathbb{C}P^k$  ( $0 < k \leq n - 2$ ), where  $0 < r < \frac{\pi}{2}$ .*

**Theorem 0.2** *Let  $M$  be a real hypersurface of  $\mathbb{C}H^n$ ,  $n \geq 2$ . If it satisfies*

$$g((A\phi - \phi A)X, Y) = 0$$

*for any vector fields  $X$  and  $Y$ , then  $M$  is locally congruent to one of the following:*

- ( $A_0$ ) a self - tube, that is, horosphere,*
- ( $A_1$ ) a geodesic hypersphere or a tube over a hyperplane  $\mathbb{C}H^{n-1}$ ,*
- ( $A_2$ ) a tube over a totally geodesic  $\mathbb{C}H^k$  ( $1 \leq k \leq n - 2$ ).*

Real hypersurfaces of type  $A_1$  and  $A_2$  in  $\mathbb{C}P^n$  and of type  $A_0$ ,  $A_1$  and  $A_2$  in  $\mathbb{C}H^n$  are said to be hypersurfaces of *type A* for simplicity.

A Jacobi field along geodesics of a given Riemannian manifold  $(M, g)$  plays an important role in the study of differential geometry. It satisfies a well known differential equation which inspires Jacobi operators. For any vector field  $X$ , the Jacobi operator is defined by  $R_X: R_X(Y) = R(Y, X)X$ , where  $R$  denotes the curvature tensor and  $Y$  is a vector field on  $M$ .  $R_X$  is a self - adjoint endomorphism in the tangent space of  $M$ , and is related to the Jacobi differential equation, which is given by  $\nabla_{\dot{\gamma}}(\nabla_{\dot{\gamma}}Y) + R(Y, \dot{\gamma})\dot{\gamma} = 0$  along a geodesic  $\gamma$  on  $M$ , where  $\dot{\gamma}$  denotes the velocity vector along  $\gamma$  on  $M$ .

In a real hypersurface  $M$  of a complex space form  $M_n(c)$ ,  $c \neq 0$ , the Jacobi operator on  $M$  with respect to the structure vector field  $\xi$ , is called the Jacobi structure operator and is denoted by  $lX = R_\xi(X) = R(X, \xi)\xi$ .

Many authors have studied real hypersurfaces from many points of view. Certain authors have studied real hypersurfaces under the condition  $\phi l = l\phi$ , equipped with one or two additional conditions. U-Hang Ki, An -Aye Lee and Seong-Baek Lee ([9]) classified real hypersurfaces in complex space forms satisfying i)  $\phi l = l\phi$  and  $A^2\xi = \theta A\xi + \tau\xi$  ( $\theta$  is a function,  $\tau$  is constant) ii)  $\phi l = l\phi$  and  $Q\xi = \sigma\xi$  (where  $Q$  is the Ricci operator,  $\sigma$  is constant). U-Hang Ki ([7]) classified real hypersurfaces in complex hyperbolic space satisfying

$\phi l = l\phi$  and  $lQ = Ql$ . U-Hang Ki with Soo Jin-Kim and Seong-Baek Lee ([8]), classified real hypersurfaces in complex space forms satisfying  $\phi l = l\phi$ ,  $lQ = Ql$ , and additional conditions on the mean curvature. U-Hang Ki, S. Nagai and R. Takagi([10]) studied real hypersurfaces in complex space forms satisfying  $\phi l = l\phi$  and  $lQ = lQ$ .

Other authors have studied real hypersurfaces under the conditions  $\nabla_X l = 0$  ( $X \in TM$ ) or  $\nabla_\xi l = 0$  ([14], [11], [5]).

In the present paper, we consider a weaker condition  $\nabla_\xi l = \mu\xi$ , where  $\mu$  is a function of class  $C^1$  on  $M$ , and classify these hypersurfaces satisfying  $\phi l = l\phi$ . Namely we prove:

**Theorem 0.3** *Let  $M$  be a real hypersurface of a complex space form  $M_n(c)$ , ( $n > 2$ ) ( $c \neq 0$ ), satisfying  $\phi l = l\phi$ . If  $\nabla_\xi l = \mu\xi$  on  $\ker(\eta)$  or on  $\text{span}\{\xi\}$ , then  $M$  is a Hopf hypersurface. Furthermore, if  $\eta(A\xi) \neq 0$ , then  $M$  locally congruent to a model space of type A.*

J. T. Cho and U - H Ki in [4] classified real hypersurfaces  $M$  of a projective space satisfying  $\phi l = l\phi$  and  $lA = Al$  on  $M$ . In the present paper we generalize this result, studying the real hypersurfaces of any complex space form satisfying  $\phi l = l\phi$  and  $lA = Al$  on the distribution on  $M$  ( $\ker(\eta)$ ) given by all vectors orthogonal to the Reeb flow  $\xi$ , or on  $\text{span}\{\xi\}$ . We prove:

**Theorem 0.4** *Let  $M$  be a real hypersurface of a complex space form  $M_n(c)$ , ( $n > 2$ ) ( $c \neq 0$ ), satisfying  $\phi l = l\phi$ . If  $lA = Al$  on  $\ker(\eta)$  or on  $\text{span}\{\xi\}$ , then  $M$  is a Hopf hypersurface. Furthermore, if  $\eta(A\xi) \neq 0$ , then  $M$  locally congruent to a model space of type A.*

For the case of  $\mathbb{C}P^n$  in order to determine real hypersurface of type A, the technical assumption  $\eta(A\xi) \neq 0$  is needed. Actually, there is a non-homogeneous tube with  $A\xi = 0$  (of radius  $\frac{\pi}{4}$ ) over a certain Kaehler submanifold in  $\mathbb{C}P^n$ , when its focal map has constant rank on  $M$  ([3]). For Hopf hypersurfaces in  $\mathbb{C}H^n$ , ( $n > 2$ ) it is known that the associated principal curvature of  $\xi$  never vanishes ([1]). However, in  $\mathbb{C}H^2$  there exists a Hopf hypersurface with  $A\xi = 0$  ([6]).

## 1 Preliminaries.

Let  $M_n$  be a Kaehlerian manifold of real dimension  $2n$ , equipped with an almost complex structure  $J$  and a Hermitian metric tensor  $G$ . Then for any

vector fields  $X$  and  $Y$  on  $M_n(c)$ , the following relations hold:

$$J^2X = -X, \quad G(JX, JY) = G(X, Y), \quad \tilde{\nabla}J = 0$$

where  $\tilde{\nabla}$  denotes the Riemannian connection of  $G$  of  $M_n$ .

Now, let  $M_{2n-1}$  be a real  $(2n-1)$ -dimensional hypersurface of  $M_n(c)$ , and denote by  $N$  a unit normal vector field on a neighborhood of a point in  $M_{2n-1}$  (from now on we shall write  $M$  instead of  $M_{2n-1}$ ). For any vector field  $X$  tangent to  $M$  we have  $JX = \phi X + \eta(X)N$ , where  $\phi X$  is the tangent component of  $JX$ ,  $\eta(X)N$  is the normal component, and

$$\xi = -JN, \quad \eta(X) = g(X, \xi), \quad g = G|_M.$$

By properties of the almost complex structure  $J$ , and the definitions of  $\eta$  and  $g$ , the following relations hold ([2]):

$$(1.1) \quad \phi^2 = -I + \eta \otimes \xi, \quad \eta \circ \phi = 0, \quad \phi\xi = 0, \quad \eta(\xi) = 1$$

$$(1.2) \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad g(X, \phi Y) = -g(\phi X, Y).$$

The above relations define an *almost contact metric structure* on  $M$  which is denoted by  $(\phi, \xi, g, \eta)$ . When an almost contact metric structure is defined on  $M$ , we can define a local orthonormal basis  $\{V_1, V_2, \dots, V_{n-1}, \phi V_1, \phi V_2, \dots, \phi V_{n-1}, \xi\}$ , called a  $\phi$ -basis. Furthermore, let  $A$  be the shape operator in the direction of  $N$ , and denote by  $\nabla$  the Riemannian connection of  $g$  on  $M$ . Then,  $A$  is symmetric and the following equations are satisfied:

$$(1.3) \quad \nabla_X \xi = \phi AX, \quad (\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi.$$

As the ambient space  $M_n(c)$  is of constant holomorphic sectional curvature  $c$ , the equations of Gauss and Godazzi are respectively given by:

$$(1.4) \quad R(X, Y)Z = \frac{c}{4}[g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y \\ - 2g(\phi X, Y)\phi Z] + g(AY, Z)AX - g(AX, Z)AY,$$

$$(1.5) \quad (\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}[\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi].$$

The tangent space  $T_p M$ , for every point  $p \in M$ , is decomposed as following:

$$T_p M = \ker(\eta)^\perp \oplus \ker(\eta)$$

where  $\ker(\eta)^\perp = \text{span}\{\xi\}$  and  $\ker(\eta)$  is defined as following:

$$\ker(\eta) = \{X \in T_p M : \eta(X) = 0\}$$

Based on the above decomposition, by virtue of (1.3), we decompose the vector field  $A\xi$  in the following way:

$$(1.6) \quad A\xi = \alpha\xi + \beta U$$

where  $\beta = |\phi \nabla_\xi \xi|$  and  $U = -\frac{1}{\beta} \phi \nabla_\xi \xi \in \ker(\eta)$ , provided that  $\beta \neq 0$ .

If the vector field  $A\xi$  is expressed as  $A\xi = \alpha\xi$ , then  $\xi$  is called a principal vector field.

Finally differentiation will be denoted by ( ). All manifolds and vector fields of this paper are assumed to be connected and of class  $C^\infty$ .

## 2 Auxiliary relations

In the study of real hypersurfaces of a complex space form  $M_n(c)$ ,  $c \neq 0$ , it is a crucial condition that the structure vector field  $\xi$  is principal. The purpose of this paragraph is to prove this condition.

Let  $V$  be the open subset of points  $p$  of  $M$ , where  $\alpha \neq 0$  in a neighborhood of  $p$  and  $V_0$  be the open subset of points  $p$  of  $M$  such that  $\alpha = 0$  in a neighborhood of  $p$ . Since  $\alpha$  is a smooth function on  $M$ , then  $V \cup V_0$  is an open and dense subset of  $M$ .

**Lemma 2.1** *Let  $M$  be a real hypersurface of a complex space form  $M_n(c)$  ( $c \neq 0$ ), satisfying  $\phi l = l\phi$  on  $\ker(\eta)$ . Then,  $\beta = 0$  on  $V_0$ .*

*Proof.* From (1.6) we have  $A\xi = \beta U$  on  $V_0$ . Then (1.4) for  $X = U$  and  $Y = Z = \xi$  yields

$$lU = \frac{c}{4}U + g(A\xi, \xi)AU - g(AU, \xi)A\xi = \frac{c}{4}U - g(U, A\xi)A\xi = (\frac{c}{4} - \beta^2)U \Rightarrow$$

$$\phi lU = (\frac{c}{4} - \beta^2)\phi U.$$

In the same way, from (1.4) for  $X = \phi U$ ,  $Y = Z = \xi$  we obtain

$$l\phi U = \frac{c}{4}\phi U.$$

The last two equations yield  $\beta = 0$ . □

**REMARK 1**

We have proved that on  $V_0$ ,  $A\xi = 0\xi$  i.e.  $\xi$  is a principal vector field on  $V_0$ . Now we define on  $V$  the set  $V'$  of points  $p$  where  $\beta \neq 0$  in a neighborhood of  $p$  and the set  $V''$  of points  $p$  where  $\beta = 0$  in a neighborhood of  $p$ . Obviously  $\xi$  is principal on  $V''$ . In what follows we study the open subset  $V'$  of  $M$  and define the following classes:

- A = hypersurfaces satisfying  $\phi l = l\phi$  and  $lA = Al$  on  $\ker(\eta)$ ,
- B = hypersurfaces satisfying  $\phi l = l\phi$  and  $lA = Al$  on  $\text{span}\{\xi\}$ ,
- C = hypersurfaces satisfying  $\phi l = l\phi$  and  $\nabla_\xi l = \mu\xi$  on  $\ker(\eta)$ ,
- D = hypersurfaces satisfying  $\phi l = l\phi$  and  $\nabla_\xi l = \mu\xi$  on  $\text{span}\{\xi\}$ .

**Lemma 2.2** *Let  $M$  be a real hypersurface of a complex space form  $M_n(c)$  ( $c \neq 0$ ), satisfying  $\phi l = l\phi$  on  $\ker(\eta)$ . Then the following relations hold on the set  $V'$  of classes A, B, C, D.*

$$(2.1) \quad AU = (\frac{\beta^2}{\alpha} - \frac{c}{4\alpha})U + \beta\xi, \quad A\phi U = -\frac{c}{4\alpha}\phi U.$$

$$(2.2) \quad \nabla_\xi \xi = \beta\phi U, \quad \nabla_U \xi = (\frac{\beta^2}{\alpha} - \frac{c}{4\alpha})\phi U, \quad \nabla_{\phi U} \xi = \frac{c}{4\alpha}U.$$

$$(2.3) \quad \nabla_\xi U = W_1, \quad \nabla_U U = W_2, \quad \nabla_{\phi U} U = W_3 - \frac{c}{4\alpha}\xi.$$

$$(2.4) \quad \nabla_\xi \phi U = \phi W_1 - \beta\xi, \quad \nabla_U \phi U = \phi W_2 + (\frac{c}{4\alpha} - \frac{\beta^2}{\alpha})\xi, \quad \nabla_{\phi U} \phi U = \phi W_3.$$

where  $W_1, W_2, W_3$  are vector fields on  $\ker(\eta)$  satisfying  $W_1, W_2, W_3 \perp U$ .

Proof. From (1.4) we get

$$(2.5) \quad lX = \frac{c}{4}[X - \eta(X)\xi] + \alpha AX - g(AX, \xi)A\xi$$

which, for  $X = U$  yields

$$(2.6) \quad lU = \frac{c}{4}U + \alpha AU - \beta A\xi.$$

The scalar products of (2.6) with  $U$  and  $\phi U$  yield respectively

$$(2.7) \quad g(AU, U) = \frac{\gamma}{\alpha} - \frac{c}{4\alpha} + \frac{\beta^2}{\alpha},$$

$$(2.8) \quad g(AU, \phi U) = \frac{1}{a}g(lU, \phi U).$$

where  $\gamma = g(lU, U) = g(\phi lU, \phi U) = g(l\phi U, \phi U)$ .

The second relation of (1.2) for  $X = U$ ,  $Y = lU$ , the condition  $\phi l = l\phi$  and the symmetry of the operator  $l$  imply:

$$g(lU, \phi U) = 0.$$

The above equation and (2.8) imply

$$(2.9) \quad g(AU, \phi U) = 0.$$

The symmetry of  $A$  and (1.6) imply

$$(2.10) \quad g(AU, \xi) = \beta.$$

From relations (2.7), (2.9) and (2.10), we obtain

$$(2.11) \quad AU = \left(\frac{\gamma}{\alpha} - \frac{c}{4\alpha} + \frac{\beta^2}{\alpha}\right)U + \beta\xi + \lambda W$$

where  $W \in \text{span}\{U, \phi U, \xi\}^\perp$  and  $\lambda = g(AU, W)$ . Combining (2.11) with (2.6) we obtain  $lU = \gamma U + \lambda\alpha W$ . Acting on this relation with the tensor

field  $\phi$  and by virtue of  $\phi l = l\phi$  we take  $l\phi U = \gamma\phi U + \lambda\alpha\phi W$ . On the other hand by virtue of (2.5) we have  $l\phi U = \frac{\epsilon}{4}\phi U + \alpha A\phi U$ . From the last two relations we obtain  $A\phi U = (\frac{\gamma}{\alpha} - \frac{c}{4\alpha})\phi U + \lambda\phi W$ .

#### On class A

Since  $lA = Al$  holds on  $\ker(\eta)$  we have  $lAW = AlW$ . This relation because of (2.5) and (2.11) implies  $\lambda\beta A\xi = 0$  and so  $\lambda = 0$ . Since  $\lambda = 0$ , equations  $lAU = AlU$ , (2.6) and (2.11) yield  $\gamma = 0$ , therefore we have the first of (2.1). Moreover from (2.5) we have  $l\phi U = \frac{\epsilon}{4}\phi U + \alpha A\phi U$  which is written as  $\phi lU = \frac{\epsilon}{4}\phi U + \alpha A\phi U$  ( $\phi l = l\phi$ ). From  $\phi lU = \frac{\epsilon}{4}\phi U + \alpha A\phi U$  and  $\gamma = \lambda = 0$  we obtain the second of (2.1). Using (1.3) for  $X \in \{\xi, U, \phi U\}$  and by virtue of (2.1) we obtain (2.2). It is well known that:

$$(2.12) \quad Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$$

Let us set  $\nabla_\xi U = W_1$  and  $\nabla_U U = W_2$ . If we use (2.2) and (2.12), it is easy to verify that  $g(\nabla_\xi U, U) = 0 = \eta(\nabla_\xi U)$  and  $g(\nabla_U U, U) = 0 = \eta(\nabla_U U)$  which means  $W_1 \perp \{\xi, U\}$  and  $W_2 \perp \{\xi, U\}$ .

On the other hand using (2.12) and the third of (2.2) we find  $\eta(\nabla_{\phi U} U) = -\frac{c}{4\alpha}$  and  $g(\nabla_{\phi U} U, U) = 0$  which means that  $\nabla_{\phi U} U$  is decomposed as  $\nabla_{\phi U} U = W_3 - \frac{c}{4\alpha}\xi$ ,  $W_3 \perp \{U, \xi\}$ . Now, by virtue of (1.3) and (2.3) for  $X = \xi, Y = U$  and  $X = Y = U$  and  $X = \phi U, Y = U$ , we get (2.4).

#### On class B

We analyze equation  $lA\xi = Al\xi$  by virtue of (1.6), (2.6) and (2.11) and we have  $\gamma U + \lambda\alpha W = 0$ . Since  $W \perp U$  we have  $\gamma = \lambda = 0$ . The rest of the proof is similar to the one in class A.

#### On class C

We have  $(\nabla_\xi l)U = \mu\xi$ . The scalar product of  $(\nabla_\xi l)U = \mu\xi$  with  $\xi$ , the symmetry of  $l$ ,  $g(lU, \phi U) = 0$  and (2.12) yield  $\mu = 0$ . In addition we have  $(\nabla_\xi l)\phi U = \mu\xi = 0$ . So, the scalar product of  $(\nabla_\xi l)\phi U = 0$  with  $\xi$ , the symmetry of  $l$  and (2.12) yield  $g(l\phi U, \phi U) = \gamma = 0$ .

Finally  $(\nabla_\xi l)\phi W = \mu\xi = 0$ . So, the scalar product of  $(\nabla_\xi l)\phi W = 0$  with  $\xi$ , the symmetry of  $l$  and (2.12) yield  $g(l\phi U, \phi W) = 0$ , which, by virtue of (2.5), the second of (2.1) and  $\gamma = 0$ , yield  $\lambda = 0$ . The rest of the proof is similar to the one in class A.

#### On class D

We analyze  $(\nabla_\xi l)\xi = \mu\xi$  and obtain  $\beta l\phi U = \mu\xi$ . But the vector fields  $l\phi U$  and  $\xi$  are linear independent, so  $l\phi U = \mu = 0$ . We analyze  $l\phi U = 0$  using (2.5) and  $A\phi U = (\frac{\gamma}{\alpha} - \frac{c}{4\alpha})\phi U + \lambda\phi W$ , and we have  $\gamma\phi U + \lambda\alpha\phi W = 0$ . This



relation and the linear independency of the vector fields  $\phi U$  and  $\phi W$  yield  $\gamma = \lambda = \mu = 0$ . The rest of the proof is similar to the one in class A.  $\square$

**Lemma 2.3** *Let  $M$  be a real hypersurface of a complex space form  $M_n(c)$  ( $c \neq 0$ ), of class A, B, C, or D. Then on  $V'$  we have  $g(\nabla_\xi U, \phi U) = -4\alpha$  and  $g(\nabla_U U, \phi U) = -4\beta + \frac{c}{4\alpha\beta}(\frac{c}{4\alpha} - \frac{\beta^2}{\alpha})$ .*

Proof. Putting  $X = U$ ,  $Y = \xi$  in (1.5), we obtain

$$(\nabla_U A)\xi - (\nabla_\xi A)U = -\frac{c}{4}\phi U.$$

Combining the last equation with (1.6), and Lemma 2.2 it follows :

$$\begin{aligned} (U\alpha)\xi + (U\beta)U + \beta W_2 + \left(-\frac{c}{4\alpha} + \frac{\beta^2}{\alpha}\right)\frac{c}{4\alpha}\phi U \\ -\xi\left(-\frac{c}{4\alpha} + \frac{\beta^2}{\alpha}\right)U - \left(-\frac{c}{4\alpha} + \frac{\beta^2}{\alpha}\right)W_1 - (\xi\beta)\xi + AW_1 = 0. \end{aligned}$$

Taking the scalar products of the last relation with  $\xi$  and  $U$  respectively, we obtain

$$(2.13) \quad (U\alpha) = (\xi\beta)$$

and

$$(2.14) \quad (U\beta) = \left(\xi\left(\frac{\beta^2}{\alpha} - \frac{c}{4\alpha}\right)\right).$$

Combining the last three equations we have

$$(2.15) \quad AW_1 = \frac{c}{4\alpha}\left(\frac{c}{4\alpha} - \frac{\beta^2}{\alpha}\right)\phi U + \left(\frac{\beta^2}{\alpha} - \frac{c}{4\alpha}\right)W_1 - \beta W_2.$$

The scalar product of (2.15) with  $\phi W_1$  yields:

$$\beta g(\phi W_1, W_2) = -g(AW_1, \phi W_1).$$

But from (2.5) we have

$$g(l\phi W_1, W_1) = g(\phi W_1, lW_1) = \alpha g(AW_1, \phi W_1).$$

Moreover  $g(l\phi W_1, W_1) = g(\phi W_1, lW_1) = -g(W_1, \phi lW_1) = -g(W_1, l\phi W_1)$  which means that

$$g(l\phi W_1, W_1) = 0.$$

The above relations lead to  $g(\phi W_1, W_2) = 0$  which, by virtue of (2.15) implies  $g(AW_1, \phi W_2) = 0$ .

In what follows we define the following functions:

$$\kappa_1 = g(W_1, \phi U) \quad \kappa_2 = g(W_2, \phi U), \quad \kappa_3 = g(W_3, \phi U).$$

Putting  $X = \phi U$ ,  $Y = \xi$  in (1.5), we obtain

$$(2.16) \quad \begin{aligned} A\phi W_1 &= \left[ \frac{3\beta c}{4\alpha} + \alpha\beta - (\phi U \alpha) \right] \xi \\ &\quad - [(\phi U \beta) + \frac{c}{4\alpha} \left( \frac{c}{4\alpha} - \frac{\beta^2}{\alpha} \right) - \beta^2] U + \frac{c}{4\alpha^2} (\xi \alpha) \phi U - \frac{c}{4\alpha} \phi W_1 - \beta W_3. \end{aligned}$$

The scalar product of (2.16) with  $\xi$  implies

$$(2.17) \quad (\phi U \alpha) = \frac{3\beta c}{4\alpha} + \alpha\beta + \kappa_1 \beta.$$

From the scalar product of (2.16) with  $U$  we get

$$\begin{aligned} g(A\phi W_1, U) &= -(\phi U \beta) - \frac{c}{4\alpha} \left( \frac{c}{4\alpha} - \frac{\beta^2}{\alpha} \right) + \beta^2 - \frac{c}{4\alpha} g(\phi W_1, U) \Rightarrow \\ g(\phi W_1, AU) &= -(\phi U \beta) - \frac{c}{4\alpha} \left( \frac{c}{4\alpha} - \frac{\beta^2}{\alpha} \right) + \beta^2 + \frac{c}{4\alpha} g(W_1, \phi U) \Rightarrow, \end{aligned}$$

which, eventually (with the aid of Lemma 2.2 and the definition of  $\kappa_1$ ) yields

$$(2.18) \quad (\phi U \beta) = \frac{c}{4\alpha} \left( \frac{\beta^2}{\alpha} - \frac{c}{4\alpha} \right) + \beta^2 + \kappa_1 \frac{\beta^2}{\alpha}.$$

The condition  $\phi lW_1 = l\phi W_1$  because of (2.5), (2.15) and (2.16) implies

$$\begin{aligned} \beta^2 \phi W_1 - \alpha \beta \phi W_2 + \alpha [(\phi U \alpha) - \frac{3\beta c}{4\alpha} - \alpha \beta] \xi + \alpha [(\phi U \beta) - \beta^2] U + \alpha \beta W_3 \\ = \kappa_1 \beta A \xi + \frac{c}{4\alpha} (\xi \alpha) \phi U. \end{aligned}$$

Taking the scalar product of the last relation with  $U$  we have

$$-2\kappa_1\beta^2 + \alpha\beta\kappa_2 + \alpha(\phi U\beta) - \alpha\beta^2 = 0.$$

If in the above relation we replace the term  $\kappa_1$  using (2.17) we obtain

$$(2.19) \quad -2\beta(\phi U\alpha) + \frac{3\beta^2c}{2\alpha} + \alpha\beta^2 + \alpha\beta\kappa_2 + \alpha(\phi U\beta) = 0.$$

The relation  $(\nabla_U A)\phi U - (\nabla_{\phi U} A)U = -\frac{c}{2}\xi$ , using Lemma 2.2 implies

$$(2.20) \quad \begin{aligned} & \frac{c}{4\alpha^2}(U\alpha)\phi U + [\frac{c}{2\alpha}(\frac{\beta^2}{\alpha} - \frac{c}{4\alpha}) + \beta^2 - (\phi U\beta)]\xi + \\ & [-\frac{3\beta c}{4\alpha} + \frac{\beta^3}{\alpha} + (\phi U(\frac{c}{4\alpha} - \frac{\beta^2}{\alpha}))]U - \frac{c}{4\alpha}\phi W_2 - A\phi W_2 \\ & + AW_3 + (\frac{c}{4\alpha} - \frac{\beta^2}{\alpha})W_3 = 0. \end{aligned}$$

The scalar product of the above relation with  $U$  yields

$$\frac{\kappa_2\beta^2}{\alpha} - \frac{3\beta c}{4\alpha} + \frac{\beta^3}{\alpha} + \phi U(\frac{c}{4\alpha} - \frac{\beta^2}{\alpha}) = 0.$$

Expanding the last relation and by virtue of (2.19) we get

$$(-\frac{3\beta^2}{\alpha^2} + \frac{c}{4\alpha^2})(\phi U\alpha) + \frac{3\beta}{\alpha}(\phi U\beta) + \frac{3\beta^3c}{2\alpha^3} + \frac{3\beta c}{4\alpha} = 0.$$

Combining the last equation with (2.17) and (2.18) we obtain  $\kappa_1 = -4\alpha$ . The scalar product of (2.15) with  $\phi U$  because of  $\kappa_1 = -4\alpha$ , yields  $\kappa_2 = -4\beta + \frac{c}{4\alpha\beta}(\frac{c}{4\alpha} - \frac{\beta^2}{\alpha})$ .  $\square$

**Lemma 2.4** *Let  $M$  be a real hypersurface of a complex space form  $M_n(c)$  ( $c \neq 0$ ), of class  $A$ ,  $B$ ,  $C$ , or  $D$ . Then the structure vector field  $\xi$  is principal on  $M$ .*

Proof. The scalar products of (2.16) and (2.20) with  $\phi U$ , yield  $(\xi\alpha) = \frac{4\alpha^2\beta}{c}\kappa_3$  and  $(U\alpha) = \frac{4\alpha\beta^2}{c}\kappa_3$ . Combining the last two relations with (2.13) and (2.14) we have

$$(2.21) \quad (\xi\alpha) = \frac{4\alpha^2\beta}{c}\kappa_3, \quad (U\alpha) = (\xi\beta) = \frac{4\alpha\beta^2}{c}\kappa_3, \quad (U\beta) = (\beta + \frac{4\beta^3}{c})\kappa_3$$

Using (1.5) for  $X = \phi W_2$ ,  $Y = \xi$  we have

$$\nabla_{\phi W_2} A\xi - A\nabla_{\phi W_2} \xi - \nabla_{\xi} A\phi W_2 + A\nabla_{\xi} \phi W_2 = \frac{c}{4} W_2,$$

which, from (1.6) is further decomposed as

$$\begin{aligned} & (\phi W_2 \alpha) \xi + \alpha \phi A \phi W_2 + (\phi W_2 \beta) U + \beta \nabla_{\phi W_2} U - \\ & A \phi A \phi W_2 - \nabla_{\xi} A \phi W_2 + A \nabla_{\xi} \phi W_2 = \frac{c}{4} W_2. \end{aligned}$$

Taking the scalar product with  $\xi$  and by using (1.6), (2.12), (2.21), Lemmas 2.2, 2.3 and  $W_1 \perp \phi W_2$  we obtain

$$(2.22) \quad (\phi W_2 \alpha) = \kappa_3 \left( \frac{16\alpha\beta^3}{c} + \beta \left( \frac{\beta^2}{\alpha} - \frac{c}{4\alpha} \right) \right).$$

On the other hand from (1.5) we get

$$\nabla_{W_3} A\xi - A\nabla_{W_3} \xi - \nabla_{\xi} A W_3 + A\nabla_{\xi} W_3 = -\frac{c}{4} \phi W_3$$

which, by virtue of (1.6) is further decomposed as

$$(W_3 \alpha) \xi + \alpha \phi A W_3 + (W_3 \beta) U + \beta \nabla_{W_3} U - A \nabla_{W_3} \xi - \nabla_{\xi} A W_3 + A \nabla_{\xi} W_3 = -\frac{c}{4} \phi W_3.$$

Taking the scalar product of the last equation with  $\xi$  and by making use of Lemma 2.2, (2.12) and (2.21) we obtain

$$(2.23) \quad (W_3 \alpha) = 3\beta \left( \frac{c}{4\alpha} - \alpha \right) \kappa_3.$$

In a similar way equation (1.5) yields  $(\nabla_{\phi W_1} A)U - (\nabla_U A)\phi W_1 = 0$ , which by virtue of Lemma 2.2 is further analyzed as

$$\begin{aligned} & (\phi W_1 \left( \frac{\beta^2}{\alpha} - \frac{c}{4\alpha} \right)) U + \left( \frac{\beta^2}{\alpha} - \frac{c}{4\alpha} \right) \nabla_{\phi W_1} U + \\ & (\phi W_1 \beta) \xi + \beta \phi A \phi W_1 - A \nabla_{\phi W_1} U - \nabla_U A \phi W_1 + A \nabla_U \phi W_1 = 0. \end{aligned}$$

The scalar product of the above equation with  $\xi$  and using  $g(\phi W_1, W_2) = 0$ , (2.21) and Lemmas 2.2, 2.3, leads to

$$(2.24) \quad (\phi W_1 \beta) = 4\alpha \kappa_3 \left( \beta + \frac{4\beta^3}{c} \right).$$

Now the calculation of Lie bracket  $[\phi U, \xi]\beta$ , by virtue of (2.18), Lemma 2.3 and (2.21), results to

$$[\phi U, \xi]\beta = \phi U(\xi\beta) + \kappa_3[-\frac{\beta c}{2\alpha} + \frac{24\alpha\beta^3}{c}].$$

On the other hand from Lemma 2.2, (2.21) and (2.24) we obtain

$$[\phi U, \xi]\beta = (\nabla_{\phi U}\xi - \nabla_{\xi}\phi U)\beta = \beta\kappa_3[\frac{c}{4\alpha} + \frac{\beta^2}{\alpha} - 4\alpha - \frac{12\alpha\beta^2}{c}].$$

Equalizing the above two relations we get

$$(2.25) \quad \phi U(\xi\beta) = \beta\kappa_3[\frac{3c}{4\alpha} + \frac{\beta^2}{\alpha} - 4\alpha - \frac{36\alpha\beta^2}{c}].$$

In a similar way, combining (2.18), (2.21), (2.22), (2.23) and Lemmas 2.2, 2.3, the Lie bracket  $[\phi U, U]\alpha$  yields

$$[\phi U, U]\alpha = \phi U(U\alpha) + 3\beta\kappa_3[\alpha + \frac{8\alpha\beta^2}{c} - \frac{c}{4\alpha}].$$

$$[\phi U, U]\alpha = (\nabla_{\phi U}U - \nabla_U\phi U)\alpha = \beta\kappa_3[\frac{c}{\alpha} - 5\alpha - \frac{12\alpha\beta^2}{c} - \frac{\beta^2}{\alpha}].$$

From the above equations we obtain

$$(2.26) \quad \phi U(U\alpha) = \beta\kappa_3[\frac{7c}{4\alpha} - 8\alpha - \frac{36\alpha\beta^2}{c} - \frac{\beta^2}{\alpha}].$$

Because of (2.13) from (2.25) and (2.26) we obtain

$$\frac{\beta}{\alpha}[c - 4\alpha^2 - 2\beta^2]\kappa_3 = 0.$$

Let us assume there is a point  $p \in V'$  such that  $\kappa_3 \neq 0$  in a neighborhood around  $p$ . Then we have  $c = 4\alpha^2 + 2\beta^2$ . Differentiating the last equation along  $\xi$  and by virtue of (2.21) and  $\kappa_3 \neq 0$  we take  $2\alpha^2 + \beta^2 = 0$  which is a contradiction. So  $\kappa_3 = 0 \Rightarrow (U\alpha) = (\xi\alpha) = 0 \Rightarrow [U, \xi]\alpha = 0$ . But the last equation, because of Lemma 2.2 yields

$$(2.27) \quad (\frac{\beta^2}{\alpha} - \frac{c}{4\alpha})(\phi U\alpha) - (W_1\alpha) = 0.$$

On the other hand from (1.5) for  $X = W_1$ ,  $Y = \xi$ , taking the scalar product with  $\xi$ , using the Lemmas 2.2, 2.3 we have  $(W_1\alpha) = \beta|W_1|^2 - \beta(4\alpha^2 + 3c)$ . The last relation, (2.27), (2.17) and Lemma 2.3 lead to

$$(2.28) \quad 12(5\alpha^2 + \beta^2)c + 64\alpha^4 = 16\alpha^2(|W_1|^2 + 3\beta^2) + 3c^2.$$

Because of (2.28):  $f(\omega) = 64\omega^2 + 60c\omega + 12c\beta^2$ , where  $\omega = \alpha^2$ , is positive for every  $\omega, \beta$ . This holds if and only if the discriminant of  $f(\omega)$  is negative for all  $\beta, c$ . But this is not true, hence we have a contradiction. Therefore  $V'$  is empty and the real hypersurface  $M$  consists only of  $V_0$  and  $V''$  i.e  $\xi$  is principal and  $M$  is a Hopf hypersurface.  $\square$

### 3 Proof of theorems

From Lemma 2.4:

$$(3.1) \quad A\xi = \alpha\xi, \quad \alpha = g(A\xi, \xi).$$

We consider a  $\phi$ -basis  $\{V_i, \phi V_i, \xi\}$ , ( $i = 1, 2, \dots, n-1$ ). From (2.5) and (3.1) we obtain

$$(3.2) \quad lX = \frac{c}{4}[X - \eta(X)\xi] + \alpha AX - \eta(X)\alpha^2\xi.$$

(3.2) for  $X = V_i$  implies

$$(3.3) \quad lV_i = \frac{c}{4}V_i + \alpha AV_i.$$

Applying  $\phi$  to (3.3) we obtain

$$(3.4) \quad \phi lV_i = \frac{c}{4}\phi V_i + \alpha\phi AV_i, \quad i = 1, \dots, n-1.$$

The relation (3.2) for  $X = \phi V_i$  yields

$$(3.5) \quad l\phi V_i = \frac{c}{4}\phi V_i + \alpha A\phi V_i.$$

Comparing (3.4) with (3.5), and by making use of the condition  $\phi l = l\phi$  we have

$$(3.6) \quad (A\phi - \phi A)V_i = 0, \quad i = 1, \dots, n-1.$$

On the other hand the action of  $\phi$  on (3.5) yields

$$(3.7) \quad \phi(l\phi V_i) = \frac{c}{4}\phi^2 V_i + \alpha\phi A\phi V_i,$$

which, by virtue of (1.1), is written in the form

$$(3.8) \quad (\phi l)\phi V_i = -\frac{c}{4}V_i + \alpha(\phi A)\phi V_i,$$

Moreover, the calculation of  $(l\phi)\phi V_i$  by virtue of (1.1) and (3.3) yields:

$$(l\phi)\phi V_i = l\phi^2 V_i = -lV_i = -\frac{c}{4}V_i - \alpha AV_i = -\frac{c}{4}V_i + \alpha A\phi^2 V_i = -\frac{c}{4}V_i + \alpha A\phi\phi V_i \Leftrightarrow$$

$$(3.9) \quad (l\phi)\phi V_i = -\frac{c}{4}V_i + \alpha(A\phi)\phi V_i$$

Comparing (3.8) and (3.9), and by making use of the condition  $\phi l = l\phi$  we have

$$(3.10) \quad (A\phi - \phi A)\phi V_i = 0$$

for every  $i = 1, \dots, n-1$ . But from (1.1) and (3.1) we also have

$$(3.11) \quad (A\phi - \phi A)\xi = 0.$$

So, (3.6), (3.10) and (3.11) imply that  $A\phi = \phi A$ . This result and the Theorems (0.1) and (0.2) complete the proof of the main Theorems.  $\square$

We must also notice that in class C (REMARK 1) we have  $(\nabla_\xi l)V_i = \mu\xi \Leftrightarrow \nabla_\xi lV_i - l\nabla_\xi V_i = \mu\xi$ , whose scalar product with  $\xi$  yields  $\mu = 0$ . Also in class D (REMARK 1) we have  $(\nabla_\xi l)\xi = \mu\xi \Rightarrow \mu = 0$ . So we have:

**Corollary 3.1** *Let  $M$  be a real hypersurface of a complex space form  $M_n(c)$  ( $c \neq 0$ ), satisfying  $\phi l = l\phi$  and  $\nabla_\xi l = \mu\xi$  on  $\ker(\eta)$  or on  $\text{span}\{\xi\}$ . Then the function  $\mu$  must be identically zero on  $M$ .*

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